- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{inv} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{inv} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., h is *absolutely integrable*.)

- An input x to a system $\mathcal{H}$ is said to be an eigenfunction of the system $\mathcal{H}$ with the eigenvalue λ if the corresponding output y is of the form

$$y = \lambda x,$$

where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Of particular interest are the eigenfunctions of LTI systems.

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system H with impulse response h,

$$H\{e^{st}\} = H(s)e^{st},$$

where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the system function (or transfer function) of the system H.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

where the a_k and s_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Part 4

Continuous-Time Fourier Series (CTFS(

- The Fourier series is a representation for *periodic* signals.
- With a Fourier series, a signal is represented as a *linear combination of complex sinusoids*.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are *eigenfunctions* of LTI systems.

Section 4.1

Fourier Series

- A set of complex sinusoids is said to be harmonically related if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{for all integer } k.$$

- The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0.
- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.
- More specifically, a linear combination of these complex sinusoids is periodic with period $T = 2\pi/\omega_0$.

- A periodic complex signal x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a (CT) Fourier series, and the c_k are called Fourier series coefficients.
- The above formula for x is often referred to as the Fourier series synthesis equation.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth harmonic components, and have the fundamental frequency $K\omega_0$.
- To denote that a signal x has the Fourier series coefficient sequence c_k, we write

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

- The periodic signal x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt,$$

where $\int_T$ denotes integration over an arbitrary interval of length T (i.e., one period of x).

- The above equation for c_k is often referred to as the Fourier series analysis equation.

- Consider the periodic signal x with the Fourier series coefficients c_k. If
- x is real, then its Fourier series can be rewritten in two other forms, known as the combined trigonometric and trigonometric forms.
- The combined trigonometric form of a Fourier series has the appearance

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

where $\theta_k = \arg c_k$.
- The trigonometric form of a Fourier series has the appearance

$$x(t) = c_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t],$$

where $\alpha_k = 2 \operatorname{Re} c_k$ and $\beta_k = -2 \operatorname{Im} c_k$.
- Note that the trigonometric forms contain only *real* quantities.

Section 4.2

Convergence Properties of Fourier Series

- Since a Fourier series can have an infinite number of terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.
- That is, when we claim that a periodic signal $x(t)$ is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?
- Consider a periodic signal x that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let x_N denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N(t)$ is equal (in some sense) to $x(t)$.

- The *error* in approximating $x(t)$ by $x_N(t)$ is given by

$$e_N(t) = x(t) - x_N(t),$$

and the corresponding *mean-squared error (MSE)* (i.e., energy of the error) is given by

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2 \, dt.$$

- If $\lim_{N\to\infty} e_N(t) = 0$ for all t (i.e., the error goes to zero at every point), the Fourier series is said to converge pointwise to $x(t)$.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be uniform.
- If $\lim_{N\to\infty} E_N = 0$ (i.e., the energy of the error goes to zero), the Fourier series is said to converge to x in the MSE sense.
- Pointwise convergence implies MSE convergence, but the converse is not true. Thus, pointwise convergence is a much stronger condition than MSE convergence.